Optimal Evolutionary Window for the Nonlinear Local Lyapunov Exponent

Xiaowei Huai[1,2], Jianping Li[3,4], Ruiqiang Ding[1,5], and Deqiang Liu[6]

[1]*State Key Laboratory of Numerical Modeling for Atmospheric Sciences and Geophysical Fluid Dynamics (LASG)*, *Institute of Atmospheric Physics*, *Beijing*, *China*

[2]*College of Earth Sciences*, *University of Chinese Academy of Sciences*, *Beijing*, *China*

[3]*State Key Laboratory of Earth Surface Processes and Resource Ecology and College of Global Change and Earth System Science (GCESS)*, *Beijing Normal University*, *Beijing*, *China*

[4]*Laboratory for Regional Oceanography and Numerical Modeling*, *Qingdao National Laboratory for Marine Science and Technology*, *Qingdao*, *China*

[5]*College of Atmospheric Sciences*, *Plateau Atmosphere and Environment Key Laboratory of Sichuan Province*, *Chengdu University of Information Technology*, *Chengdu*, *China*

[6]*Fujian Meteorological Observatory*, *Fuzhou*, *China*

Abstract

The impact of the length of the evolutionary window (EW) on the estimation of the predictability limit of the Lorenz-63 model using the nonlinear local Lyapunov exponent (NLLE) method is studied. The structure of the initial errors and error growth dynamics are analyzed. It is found that there exists an optimal EW, at which the estimated predictability limit is closest to its theoretical value. With a shorter EW, the predictability limit is underestimated, while at longer EWs it is overestimated. The optimal EW is approximately equal to the decorrelation time of the system. A preliminary explanation for this link, based on the loss of information from the initial state, is given.

(**Citation**: Huai, X., J. Li, R. Ding, and D. Liu, 2017: Optimal evolutionary window for the nonlinear local Lyapunov exponent. *SOLA*, **13**, 125−129, doi:10.2151/sola.2017-023.)

1. Introduction

The field of predictability has been extensively studied, beginning with the pioneering work of Thompson (1957). In particular, the idea of limits of predictability is considered to be of great importance (Lorenz 1963; Chou 1989). Three approaches were proposed for investigating this problem, i.e. the empirical approach, the dynamical approach, and the dynamical-empirical approach (Lorenz 1969b). The first is based upon the occurrence of natural analogues and use them to make forecast about the future events (Lorenz 1969a; Gutzler and Shukla 1984). The second is based upon equations of the atmosphere and is referred as the numerical weather prediction (Smagorinsky, 1963; Lorenz 1982). The last approach "is partly dynamical and partly empirical" as noted by Lorenz (1969b) and takes errors of different scales into account (1969c). Up to now, most studies concerning the limit of predictability are based upon models (Lorenz 1965; Leith 1983; Simmons et al. 1995; Mu et al. 2002). Recently, several studies have shown that the limit of predictability achieved through models would depend on the calculation, the accuracy of computers and the numerical model itself (Feng et al. 2001; Li et al. 2000; Li et al. 2001; Li et al. 2006)

In view of the limitations of quantifying the predictability limit by models as mentioned above, a theoretical approach based upon the nonlinear error growth dynamics was introduced. The so-called nonlinear local Lyapunov exponent (NLLE) method has been shown to be efficient at estimating predictability limits of chaotic systems. (Chen et al. 2005; Ding and Li 2007; Ding et al. 2008a, 2008b; Li and Wang, 2008; Li and Ding 2011, 2013, 2015; Ding et al. 2015). The value of the NLLE depends on the initial condition in phase space, the magnitude of initial errors and the time. For chaotic systems whose governing equations are explicitly known, it is possible to simply integrate two analogous initial states forward in time and compare their trajectories.

However, in practice, the governing equations are unknown for chaotic systems. Often only one or several variables of the system are available for observations. In such cases, the dynamics of the chaotic system are hard to retrieve. Alternatively, some turn to the natural analogs, which mean that two states resemble each other in the atmospheric observational datasets. By observing the trajectory of the analogous point, we can gain knowledge about the trajectory of the referential point, thus the rate of error growth can be determined as long as the analogous trajectory mimics the referential trajectory (Lorenz 1969a; Gutzler and Shukla 1984). Van den Dool (1994) estimated that good-quality analogs could be calculated from current data libraries, if the analogs were restricted to a small area and a small number of degrees of freedom. In view of this, a local dynamical analog (LDA) method was introduced for calculating the NLLE from experimental or observational time series (Li and Ding 2011). The methodology for LDA is based upon the assumption that two states can be considered to be analogous if their evolutionary trajectories are similar over a short period of time, which is referred to as an evolutionary window (EW). In other words, EW is a time interval during which the evolutionary distance between two states are calculated. Hence, not only the initial information but also the evolutionary information of the referential point and the analogous point are taken into consideration. Only when the sum of the initial distance and the evolutionary distance is minimized over all sampling points can we term this specific point as a LDA of the referential point. The algorithm of the LDA is applied to the Lorenz-63 model as well as real atmosphere. Li and Ding (2011) investigated the temporal-spatial distribution of atmospheric predictability limit by the LDA method. Ding et al. (2015) estimated the limit of decadal-scale climate predictability using observational data.

By definition, the accumulated error between the referential trajectory and the analogous trajectory is a function of the EW. This implies that the choice of EW will be an important factor in determining the success of LDA. There are three more questions to be further explored and have not yet been discussed in previous studies. First, what influence does the EW have on the initial error and the error growth of the LDA? Second, what is the connection between the EW and the predictability limit calculated using NLLE? Finally, how should the correct EW be selected such that the predictability limit estimated is close to the theoretical value? This paper aims to investigate these important questions so as to further improve the NLLE estimation method.

The remainder of the paper is arranged as follows. A brief

Corresponding author: Jianping Li, College of Global Change and Earth System Science (GCESS), Beijing Normal University, Xueyuan South Road, Haidian District, Beijing 100875 China. E-mail: ljp@bnu.edu.cn.

description of the LDA method and the basic setup of our numerical experiment are given in Section 2. Section 3 presents the results of a quantitative comparison of different EWs using the Lorenz-63 system, followed by some preliminary interpretations. Finally, a summary is presented in Section 4.

2. Model and methods

2.1 Lorenz-63 model

The Lorenz-63 model (Lorenz 1963) can be expressed as

$$\begin{cases} \dot{x} = -\sigma(x-y) \\ \dot{y} = -xz + rx - y, \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

where the parameters used are $\sigma = 10$, $r = 28$, and $b = 8/3$. The integration scheme used here is the fourth-order Runge–Kutta method, with a stepsize $h = 0.01$. We use the Lorenz-63 model so that error growth can be calculated theoretically as well as estimated numerically. Also, the Lorenz-63 model is simple and well-studied.

2.2 LDA algorithm

For a chaotic system, often we cannot observe all the variables of it. In many cases, only one or several variables are available for observation. How can we estimate the predictability limit based on a single time series? That's what the NLLE is for. What we are trying to explore in our manuscript is how to improve the NLLE in estimating the predictability limit given only one or several variables are available for observations for a chaotic dynamical system. Suppose we can observe variable x from an n-dimensional system denoted by the time series $\{x(t_j), j = 0, 1, 2 \ldots m-1\}$, where m is the length in time steps of the series. We now describe an algorithm for finding local analogs, following Li and Ding (2011).

For $x(t_0)$, a reference point at time t_0, we seek its LDA, denoted by $x(t_k)$, from the raw series $\{x(t_j), j = 0, 1, 2 \ldots m-1\}$. Besides the initial distance between two points, the distance after an evolutionary period is also considered. For a random point $x(t_j)$ ($|t_j - t_0| > t_d$, where t_d is the decorrelation time, or the time required for the autocorrelation of the time series to drop to zero to ensure that a good analog found is not purely due to persistence), the initial distance is given by (see Fig. 1):

$$d_0 = |x(t_j) - x(t_0)|. \tag{2}$$

We assume that two points are analogous if they evolve in a similar manner over a short time interval $\tau = K * \Delta$, where Δ is the sampling interval of the time series (i.e., $\Delta = t_i - t_{i-1}$) and Δ equals 0.01. K is the number of sampling intervals. The time interval τ is an EW as previously defined. Within this certain EW, the evolutionary distance d_e between the two points $x(t_0)$ and $x(t_j)$ is defined as

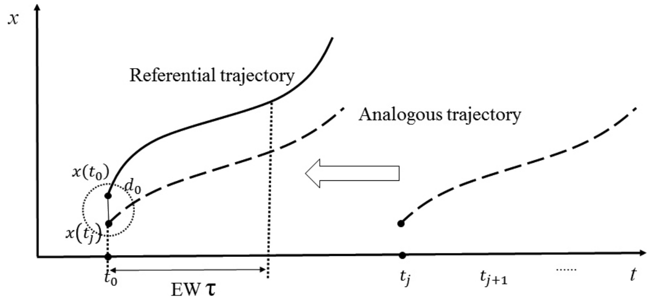

Fig. 1. A schematic representation of the LDA method. The referential trajectory (solid line) starts from time t_0. An analogous trajectory (dashed line) starting from time t_j is checked with an evolutionary window τ.

$$d_e = \sqrt{\frac{1}{K+1}\sum_{i=0}^{K}[x(t_i) - x(t_{j+i})]^2}. \tag{3}$$

The total distance then includes not only the initial separation but also the evolved separation distance, i.e.,

$$d_t = d_0 + d_e. \tag{4}$$

Therefore, d_t can be considered to be a good metric by which to find the LDA in phase space of each reference point, that means, given a specific EW τ, we check each point in the time series $\{x(t_j), j = 0, 1, 2 \ldots m-1\}$ to be its LDA by locating the minimum value of d_t. Suppose d_t acquires its minimum value at $t = t_k$, $x(t_k)$ is said to be the LDA of $x(t_0)$. To conclude, the LDA of $x(t_0)$ is defined by a point $x(t_k)$ of which d_t has the minimum value over all sampling points.

Of course, this approach may not exclude points for which x and the most closely related variables remain close in value while other variables evolve rather differently, a situation that is especially relevant in high-dimensional chaotic systems. Therefore, the analogs calculated based on the variable x can be regarded only as local analogs. However, by considering both initial distance and evolved distance, we will demonstrate here whether the method is capable of finding true analogs across all directions in phase space, especially when using the appropriate EW t_k, as shown in Section 3.

2.3 Experiment design

We used $N = 5 \times 10^4$ points in our experiment. For each reference point, we found its LDA using the algorithm described above. Integrating the Lorenz-63 model from each reference point, the geometric mean error (GME) can be calculated at each timestep l ($l = 0, \ldots, 3000$) as

$$\mathrm{GME}(l) = \sqrt[N]{\prod_{p=1}^{N}|f_p(l) - g_p(l)|}, \tag{5}$$

where f_p and g_p are the reference trajectory and its analogous trajectory at each point, respectively, $p = 1, \ldots, N$. The use of GME in error dynamics is thoroughly discussed by Ding and Li (2011). We calculate GME curves by incrementing K from 0 to 1200 in steps of 10. Thus the EW ranges from 0 to 12 by 0.1. The predictability limit is then estimated as 85% of the saturation error level in order to remove the problem of small fluctuations in the saturated error (Ding and Li 2011). All of the following calculations are based on the time series of variable x, unless explicitly stated.

3. Results and discussion

From the algorithm stated above, we know that the value of an EW is crucial to the LDA method. To illustrate this, we first measure the impact of an EW on the magnitude of the initial error between the reference point and its LDA. Here, the initial error of each variable is defined as the absolute difference between the referential point and its LDA, while the total error is the sum of initial errors in all variables.

When the EW length is set to zero, only Eq. (2) is applied to find the LDA. Since there is no evolutionary distance included, we would have $d_t = d_0$. Without considering the evolutionary information, the projection of the analog state on the x-axis being close to the reference state does not guarantee that initial errors will also be small in other directions. In fact, initial errors on the y- and z-axis are several magnitudes larger (Figs. 2b and 2c), and so the total errors are also quite large (Fig. 2d). With EW equal to zero, the method is thus liable to produce false analogs. As EW is increased, the initial errors on the x-axis become larger, but still are relatively small compared with its standard deviation (7.9 for variable x) (Fig. 2a). Meanwhile, the initial errors in other directions are reduced dramatically and become much smaller than their individual standard deviations. The total errors are now much smaller, indicating that the method is finding true analogs

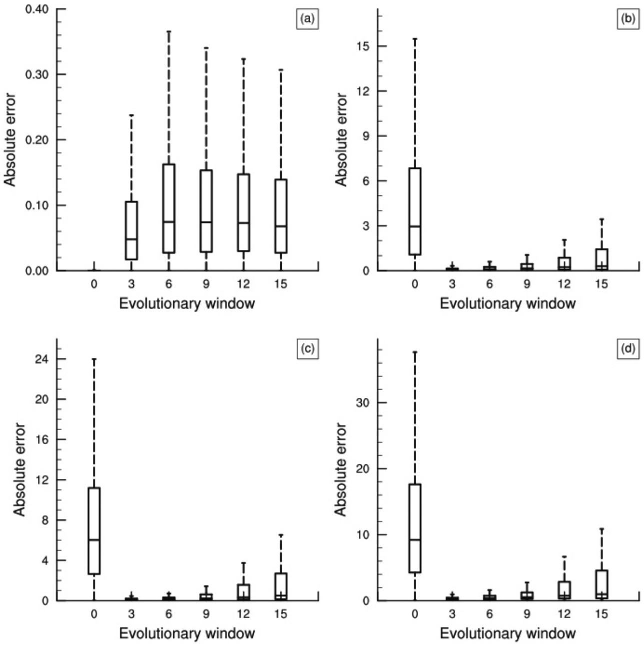

Fig. 2. Box plot of absolute error in the variables x (a), y (b), and z (c), and the total error (d) in the Lorenz-63 system, for different EW lengths. The upper whiskers extend to 1.5 times the interquartile range above the upper quartile.

(Fig. 2d). However, the distribution of the initial errors varies with EW. Figure 2d shows that the distribution of initial errors for all points used is become increasingly spread as EW increases from 0. There appears to exist an optimal value of EW between 0 and 3 for which the initial errors are least spread.

Examination of error growth dynamics can provide a clear view of the impact of EW on LDA. Figure 3 shows the GME of the variable x for several representative values of EW, along with the theoretical error evolution for an initial error of magnitude 10^{-2}. The choose of initial error is based on the fact that initial distance for variables x, y, and z between the reference point and is LDA has their maximum probability distributions at the magnitude of 0.01 (not shown). The theoretical error growth equations are based on Equation (25) of Li and Ding (2011), from which the practical predictability limit can be acquired. All the curves converge to the same saturation level (Wang et al. 2012), and show almost the same error growth rates in the nonlinear error growth period. However, error growth rates vary considerably with EW

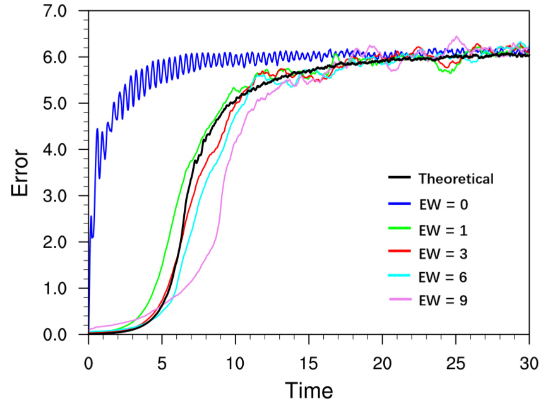

Fig. 3. GME of the variable x as estimated by the LDA method using different values of EW (colors), and the theoretical error of x, calculated from the Lorenz-63 error growth equations for an initial perturbation of magnitude of 10^{-2} (black).

in the linear error growth period. For an EW of zero, the error grows most rapidly, which is likely to lead to a shorter estimated predictability limit. As EW increases to 3, the error growth rate at the beginning of the integration period is almost identical to the theoretical one though a slight deviation after time exceeds 6. Afterwards, the error growth rate in the initial period falls below the theoretical value when EW becomes larger than 3. As a result, the predictability limit will be overestimated compared with its theoretical value.

From the initial errors and the error growth dynamics, it's easy to conjecture that there is a specific EW where the ratio of good analogs is the highest than other EWs. With this EW used in the LDA method, the estimated predictability limit would be closest to that calculated from the error growth equations. In fact, the accumulated differences between GME and the theoretical error from time 0 to 30 form a quadratic function of EW (Fig. 4b). The function possesses a minimum at a time slightly less than 3. This minimum value is always the same regardless of the accumulated period (Fig. 4a). In fact, we tested several accumulated time, and the minimum values stay the same (figure not shown). The EW corresponding to the minimum value of accumulated error is a unique property of the LDA method and is termed as the optimal evolutionary window (OEW).

The estimated predictability limits are shown as a function of EW in Fig. 5. The point of intersection between the estimated predictability limit curve and the practical predictability limit corresponds to the OEW. Below (above) this point, the predictability limit is underestimated (overestimated). Nevertheless, the

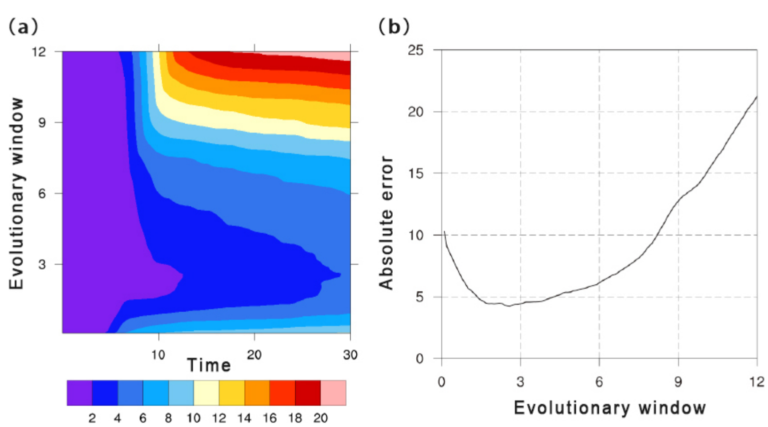

Fig. 4. (a) Accumulated absolute difference between GME and the theoretical error as a function of time and EW. (b) Accumulated absolute difference at time 30 as a function of EW.

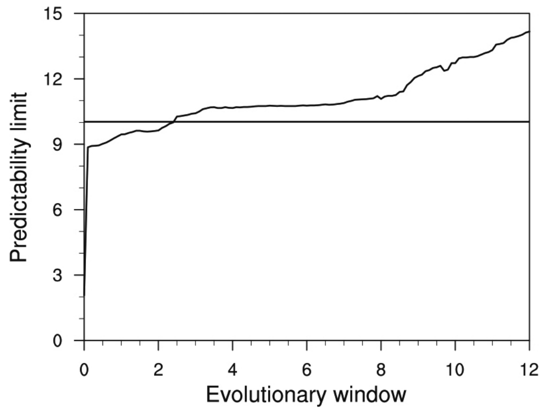

Fig. 5. Estimated predictability limit of x as a function of EW (solid curve). Predictability limit is defined as the time at which the error reaches 85% of its saturation level. The solid horizontal line indicates the predictability limit calculated from the theoretical error, and the dashed vertical line highlights the point of intersection between the estimated and practical limits.

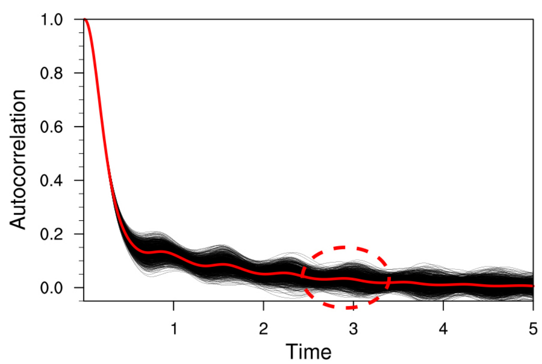

Fig. 6. Autocorrelogram for the Lorenz-63 model, from 1000 different random initial points (black lines) and their average (red solid line). The red dashed circle indicates the vicinity of the OEW.

estimated predictability limit is in reasonable agreement with the practical predictability limit when EW is close to OEW.

To conclude, the OEW is a specific time interval for calculating the evolutionary distance in the LDA method by which the error growth rate is most similar to the real one. With OEW used in the NLLE method, the accumulated error from the theoretical error growth is smaller, the ratio of good analogs is higher and the estimated predictability is more accurate. However, the value of OEW is yet to be explained. Previous studies have shown that the influence of the initial field decays over time (Li and Chou 1997). Information from the initial state can be formally considered to be lost when the autocorrelation of the series drops to zero. In the LDA method, we improve the collinearity between integral vectors by minimizing the difference between the reference and analogous trajectories. As EW is increased, nonlinearity begins to dominate the error growth, which disrupts the LDA method. When EW exceeds the decorrelation time, there is no true signal from the initial state remaining from which to identify analogs. On the contrary, the noise included will make it harder to find good analogs. It may be concluded, therefore, that the decorrelation time is approximately equal to the OEW. To confirm this, the autocorrelogram for the Lorenz-63 model, using 1000 different random initial states, is plotted in Fig. 6. The autocorrelation drops rapidly between times of 0 and 0.5. The rate of decay then decreases, until the autocorrelation finally reaches zero at a time of around the length of the OEW. After this point, information from the initial field is totally lost.

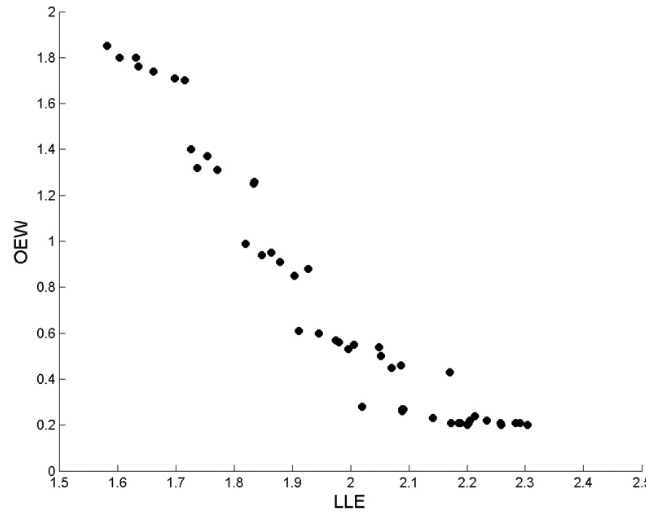

Fig. 7. Scatter plot of the OEW and the LLE of the Lorenz-63 system with parameter r ranges from 51 to 100 by 1. Parameters $\sigma = 16$, $b = 4.0$.

4. Conclusions

In this paper, the impact of EW on estimation of the predictability limit using NLLE has been studied. As EW is increased from zero, the distribution of the initial errors becomes less spread. More analogs are good analogs among all the reference points used. Thus the error growth rates based on the LDA method converge to the theoretical growth rate. Eventually, the estimated predictability limit approached to the accurate practical predictability limit. However, above a certain value of EW, the distribution of the initial errors becomes more spread. The ratio of good analogs decrease, which leads to the error growth curves diverge from the theoretical error growth curve. As a result, the predictability limit is overestimated. Between them, there exists an OEW for the NLLE method for which the error growth dynamics most closely follow theoretical calculations. At the OEW, the ratio of good analogs is higher which leads to a closer of error growth to the theoretical one. Ultimately, the predictability limit is estimated most accurately. For the Lorenz-63 system, this OEW is the time at which the autocorrelation falls to zero. As can be seen that the estimated predictability limit is close to the theoretical value whenever EW is fairly close to the OEW. So long as the EW used is close to OEW, further refinement of the OEW estimate will not be necessary.

Further investigation shows that the OEW may be related to the largest Lyapunov exponent (LLE) of the chaotic systems. By comparing the LLE with the OEW of the Lorenz-63 system with parameters $\sigma = 16$, $b = 4.0$ and parameter r varies from 51 to 100 by 1, we conclude that the larger the LLE, the shorter the OEW (Fig. 7). The reason behind this is that with a larger LLE, the divergent rate of initial filed increases, thus lead to a shorter autocorrelation time and thus a shorter OEW.

More work remains to be done, including repeating the study using other chaotic dynamical models and eventually, the real atmosphere. Also, the LDA method should be more widely exploited. We plan to carry out such a study in future work.

Acknowledgments

The authors are very grateful to Zhaolu Hou for valuable discussions. We also thank three anonymous reviewers for their valuable comments that helped to improve the manuscript a lot. This research was supported by the National Natural Science Foundation of China (41375110,41522502), and SOA International Cooperation Program on Global Change and Air-Sea Interactions

(GASI-IPOVAI-03).

Edited by: J. Ruiz

References

Chou, J. F., 1989: Predictability of the atmosphere. *Adv. Atmos. Sci.*, **6**, 335−346.

Chen, B. H., J. P. Li, and R. Q. Ding, 2006: Nonlinear local Lyapunov exponent and atmospheric predictability research. *Sci. China, Ser. D*, **49**, 1111−1120.

Ding, R. Q., and J. P. Li, 2007: Nonlinear finite-time Lyapunov exponent and predictability. *Phys. Lett. A*, **364**, 396−400.

Ding, R. Q., and J. P. Li, 2011: Comparisons of two ensemble mean methods in measuring the average error growth and the predictability. *Acta. Meteor. Sin.*, **25**, 395−404.

Ding, R. Q., J. P. Li, and K.-J. Ha, 2008a: Trends and interdecadal changes of weather predictability during 1950s−1990s. *J. Geophys. Res.*, **113**, D24112, doi:10.1029/2008JD010404.

Ding, R. Q., J. P. Li, and K. J. Ha, 2008b: Nonlinear local Lyapunov exponent and quantification of local predictability. *Chinese Phys. Lett.*, **25**, 1919−1922.

Ding, R. Q., J. P. Li, F. Zheng, J. Feng, and D. Q. Liu, 2015: Estimating the limit of decadal-scale climate predictability using observational data. *Climate Dyn.*, **46**, 1563−1580.

Feng, G. L., X. G. Dai, A. H. Wang, and co-authors, 2001: The study of the predictabil-ity in chaotic systems. *Chin. Phys.*, **50**, 606−611 (in Chinese).

Feng, J., R. Q. Ding, D. Q. Liu, and J. P. Li, 2014: The application of nonlinear local Lyapunov vectors to ensemble predictions in the Lorenz systems. *J. Atmos. Sci.*, **71**, 3554−3567.

Gutzler, D. S., and J. Shukla, 1984: Analogs in the wintertime 500 mb height field. *J. Atmos. Sci.*, **41**, 177−189.

Leith, C. E., 1983: Predictability in theory and practice. *Large-Scale Dynamics Processes in the Atmosphere*. Hoskins, B., and R. Pearce, Eds., Academic Press, 391−405.

Li, J. P., and J. F. Chou, 1997: Existence of the atmosphere attractor. *Science in China Series D: Earth Sciences*, **40**, 215−220.

Li, J. P., Q. C. Zeng, and J. F. Chou, 2000: Computational uncertainty principle in nonlinear ordinary differential equations I. numerical results. *Sci. China (Ser. E)*, **43**, 449−460.

Li, J. P., Q. C. Zeng, and J. F. Chou, 2001: Computational uncertainty principle in nonlinear ordinary differential equations II. theoretical analy-sis. *Sci. China (Ser. E)*, **44**, 55−74.

Li, J. P., R. Q. Ding, and B. H. Chen, 2006: Review and prospect on the predictability study of the atmosphere. *Review and Prospects of the Developments of Atmosphere Sciences in Early 21st Century*, China Meteorology Press, 96−104.

Li, J. P., and R. Q. Ding, 2011: Temporal–spatial distribution of atmospheric predictability limit by local dynamical analogs. *Mon. Wea. Rev.*, **139**, 3265−3283.

Li, J. P., and R. Q. Ding, 2013: Temporal–spatial distribution of the predictability limit of monthly sea surface temperature in the global oceans. *Int. J. Climatol.*, **33**, 1936−1947, doi: 10.1002/joc.3562.

Li, J. P., and R. Q. Ding, 2015: Seasonal and interannual weather prediction. In: North G., *Encyclopedia of Atmospheric Sciences (2nd edition)*, J. Pyle and F. Zhang, Eds., Academic Press and Elsevier, **6**, 303−312.

Li, J. P., and S. Wang, 2008: Some mathematical and numerical issues in geophysical fluid dynamics and climate dynamics. *Commun. Comput. Phys.*, **3**, 759−793.

Lorenz, E. N., 1963: Deterministic Nonperiodic Flow. *J. Atmos. Sci.*, **20**, 130−141.

Lorenz, E. N., 1965: A study of the predictability of a 28-variable atmospheric model. *Tellus*, **17**, 321−333.

Lorenz, E. N., 1969a: Atmospheric Predictability as Revealed by Naturally Occurring Analogues. *J. Atmos. Sci.*, **26**, 636−646.

Lorenz, E. N., 1969b: Three Approaches to Atmospheric Predictability. *Bull. Amer. Meteor. Soc.*, **50**, 345−349.

Lorenz, E. N., 1969c: The predictability of a flow which possesses many scales of motion. *Tellus*, **21**, 289−307.

Lorenz, E N., 1982: Atmospheric predictability experiments with a large numerical model. *Tellus*, **34**, 505−513.

Lorenz, E. N., 1996: Predictability: A problem partly solved. *Proceeding of Seminar on Predictability*, Vol. I, Reading, United Kindom, ECMWF, 1−18.

Mu, M., W. S. Duan, and J. C. Wang, 2002: The predictability problems in numerical weather and climate prediction. *Adv. Atmos. Sci.*, **19**, 191−204.

Simmons, A. J., R. Mureau, and T. Petroliagis, 1995: Error growth and estimates of predictability from the ECMWF forecasting system. *Quart. J. Roy. Meteor. Soc.*, **121**, 1739−1771.

Smagorinsky, J., 1963: General circulation experiments with the primitive equations: I. The basic experiment. *Mon. Wea. Rev.*, **91**, 99−164.

Thompson, P. D., 1957: Uncertainty of initial state as a factor in the predictability of large scale atmospheric flow patterns. *Tellus*, **9**, 275−295.

Van den Dool, H. M., 1994: Searching for analogues, how long must we wait? *Tellus A*, **46**, 314−324.

Wang, P. F., J. P. Li, R. Q. Ding, and co-authors, 2012: Studies of the attractor property of error equations for the Lorenz system. *Climatic Environ. Res.*, **17**, 574−582 (in Chinese).

Manuscript received 24 March 2017, *accepted 1 June 2017*
SOLA: *https://www.jstage.jst.go.jp/browse/sola/*